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Brauer indecomposability of Scott modules and local subgroups

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1. Introduction

Let p be a prime number and k an algebraically closed field of characteristic p. For a p-subgroup Q of a finite group G and a kG-module M, the Brauer quotient $M(Q)$ of M with respect to Q is naturally a $kN_G(Q)$-module. A kG-module M is said to be *Brauer indecomposable* if $M(Q)$ is indecomposable or zero as a $kC_G(Q)$-module for any p-subgroup Q of G ([6]). Brauer indecomposability of p-permutation modules is important for constructing stable equivalences of Morita type between blocks of finite groups (see [2]).

There is a connection between Brauer indecomposability of p-permutation kG-modules and fusion systems, as shown in [6]. The main result in [6] is the following.

Theorem 1 ([6, Theorem 1.1]). *Let P be a p-subgroup of G and M an indecomposable p-permutation kG-module with vertex P. If M is Brauer indecomposable, then $\mathcal{F}_P(G)$ is a saturated fusion system.*

In the case that P is abelian and M is the Scott kG-module $S(G,P)$, it is known that the converse of the above theorem holds.

Theorem 2 ([6, Theorem 1.2]). *Let P be an abelian p-subgroup of G. If $\mathcal{F}_P(G)$ is saturated, then $S(G,P)$ is Brauer indecomposable.*

In general, the above theorem does not holds in the case that P is non-abelian. However, there are some cases in which the Scott kG-module $S(G,P)$ is Brauer indecomposable for non-abelian P (see [5, 7]). Moreover, it was shown that there are some relationships between Brauer indecomposability of Scott modules and fusion systems ([3, 5]). In particular, we proved the following theorem in [3].

Theorem 3 ([3, Theroem 1.3]). *Let G be a finite group and P a p-subgroup of G. Suppose that $M = S(G,P)$ and that $\mathcal{F}_P(G)$ is saturated. Then the following are equivalent.*

(i) *M is Brauer indecomposable.*

(ii) *$\mathrm{Res}^{N_G(Q)}_{QC_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable for each fully normalized subgroup Q of P.*

If these conditions are satisfied, then $M(Q) \cong S(N_G(Q), N_P(Q))$ for each fully normalized subgroup $Q \leq P$.

The above theorem gives a criterion to determine whether the Scott module $S(G,P)$ is Brauer indecomposable.

We investigate the possibility of providing applications of the above theorem. In this paper, we will prove the following result.

Theorem 4. *Let G be a finite group and P a p-subgroup of G. Suppose that $\mathcal{F} := \mathcal{F}_P(G)$ is a saturated fusion system. Consider the following two conditions:*
 (i) *$S(N_G(Q), N_P(Q))$ is Brauer indecomposable for each fully $\mathcal{F}$-normalized subgroup $Q \leq P$.*
 (ii) *$S(G,P)$ is Brauer indecomposable.*
Then (i) implies (ii), and the converse holds if $\mathcal{F} = \mathcal{F}_P(N_G(P))$.

The above theorem shows that there exists some relationship between G and its local subgroups in terms of the Brauer indecomposability of Scott modules, and will be a useful tool for the study of the Brauer indecomposability of Scott modules.

2. Preliminaries

2.1. **Scott modules.** First, We recall the definition of Scott modules and some of its properties:

Definition 5. For a subgroup H of G, the *Scott kG-module $S(G,H)$ with respect to H* is the unique indecomposable summand M of $\mathrm{Ind}_H^G k_H$ such that $k_G \mid M$.

If P is a Sylow p-subgroup of H, then $S(G,H)$ is isomorphic to $S(G,P)$. By definition, the Scott kG-module $S(G,P)$ is a p-permutation kG-module.

By Green's indecomposability criterion, the following result holds.

Lemma 6. *Let H be a subgroup of G such that $|G : H| = p^a$ (for some $a \geq 0$). Then $\mathrm{Ind}_H^G k_H$ is indecomposable. In particular, we have that*

$$S(G,H) \cong \mathrm{Ind}_H^G.$$

The following theorem gives us information of restrictions of Scott modules.

Theorem 7 ([4, Theorem 1.7])**.** *Let P be a p-subgroup of H. Let Q be a maximal element of $P \cap_G H = \{{}^gP \cap H \mid g \in G\}$. Then $S(H,Q)$ is a direct summand of $\mathrm{Res}_H^G S(G,P)$.*

2.2. **Brauer quotients.** Let M be a kG-module and H a subgroup of G. We denote by M^H the set of H-fixed elements in M. For subgroups L of H, we denote by Tr_H^G the trace map $\mathrm{Tr}_L^H : M^L \longrightarrow M^H$. Brauer quotients are defined as follows.

Definition 8. Let M be a kG-module. For a p-subgroup Q of G, the Brauer quotient of M with respect to Q is the k-vector space

$$M(Q) := M^Q / (\sum_{R<Q} \mathrm{Tr}_R^Q(M^R)).$$

This k-vector space has a natural structure of $kN_G(Q)$-module.

Brauer quotients have the following well-known properties.

Proposition 9. *Let P be a p-subgroup of G and $M = S(G,P)$. Then $M(P) \cong S(N_G(P),P)$.*

Proposition 10. *Let M be an indecomposable p-permutation kG-module with vertex P. Let Q be a p-subgroup of G. Then $Q \leq_G P$ if and only if $M(Q) \neq 0$.*

2.3. **Fusion systems.** For subgroups Q, R of G, we denote by $\mathrm{Hom}_G(Q, R)$ the set of all group homomorphisms from Q to R which are induced by conjugation in G. For a p-subgroup P of G, the *fusion system* $\mathcal{F}_P(G)$ of G over P is the category whose objects are the subgroups of P and whose morphism set from Q to R is $\mathrm{Hom}_G(Q, R)$. We refer the reader to [1] for background involving fusion systems.

Definition 11. Let P be a p-subgroup of G

(i) A subgroup Q of P is said to be *fully normalized* in $\mathcal{F}_P(G)$ if $|N_P({}^xQ)| \leq |N_P(Q)|$ for all $x \in G$ such that ${}^xQ \leq P$.

(ii) A subgroup Q of P is said to be *fully automized* in $\mathcal{F}_P(G)$ if $p \nmid |N_G(Q) : N_P(Q)C_G(Q)|$.

(iii) A subgroup Q of P is said to be *receptive* in $\mathcal{F}_P(G)$ if it has the following property: for each $R \leq P$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}_P(G)}(R, Q)$, if we set

$$N_\varphi := \{g \in N_P(Q) \mid \exists h \in N_P(R), c_g \circ \varphi = \varphi \circ c_h\},$$

then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}_P(G)}(N_\varphi, P)$ such that $\overline{\varphi}\,|_R = \varphi$.

Saturated fusion systems are defined as follows.

Definition 12. Let P be a p-subgroup of G. The fusion system $\mathcal{F}_P(G)$ is *saturated* if the following two conditions are satisfied:

(i) P is fully normalized in $\mathcal{F}_P(G)$.

(ii) For each subgroup Q of P, if Q is fully normalized in $\mathcal{F}_P(G)$, then Q is receptive in $\mathcal{F}_P(G)$.

For example, if P is a Sylow p-subgroup of G, then $\mathcal{F}_P(G)$ is saturated.

3. Proof of Theorem 4

In this section, we give a proof of Theorem 4.

For a saturated fusion system $\mathcal{F}$ over p-group P and a subgroup Q of P, the normalizer fusion system $N_{\mathcal{F}}(Q)$ of Q is defined and is a fusion system over $N_P(Q)$ (see [1, II, §2]). We note that if $\mathcal{F} = \mathcal{F}_P(G)$, then $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(N_G(Q))$.

Proof of Theorem 4. Suppose that (i) holds. Let Q be a fully $\mathcal{F}$-normalized subgroup of P. Then $S(N_G(Q), N_P(Q))(Q)$ is indecomposable, and we have that

$$S(N_G(Q), N_P(Q)) \cong S(N_G(Q), N_P(Q))(Q).$$

Therefore, $S(G, P)$ is Brauer indecomposable by Theorem 3.

Next, suppose that (ii) and $\mathcal{F} = \mathcal{F}_P(N_G(P))$ hold. Then any subgroup Q of P is fully $\mathcal{F}$-normalized. Let Q be any subgroup of P. Then $\mathcal{F}_{N_P(Q)}(N_G(Q)) = N_{\mathcal{F}}(Q)$ is saturated by [1, II, Theorem 2.1]. Let R be a fully $N_{\mathcal{F}}(Q)$-normalized subgroup of $N_P(Q)$. It is sufficient to show that $S(N_{N_G(Q)}(R), N_{N_P(Q)}(R))$ is indecomposable as $kC_{N_G(Q)}(R)$-module by Theorem 3.

Since QR is fully $\mathcal{F}$-normalized, $S(N_G(QR), N_P(QR))$ is indecomposable as $kC_G(QR)$-module, and hence is also indecomposable as $kC_{N_G(Q)}(R)$-module. Therefore, it is sufficient to show that

$$\mathrm{Res}^{N_G(QR)}_{N_{N_G(Q)}(R)} S(N_G(QR), N_P(QR)) \cong S(N_{N_G(Q)}(R), N_{N_P(Q)}(R)),$$

and if we show that $N_{N_P(Q)}(R)$ is a maximal element of $N_P(QR) \cap_{N_G(QR)} N_{N_G(Q)}(R)$, then the isomorphism holds by Theorem 7 and the indecomposability of $S(N_G(QR), N_P(QR))$ as a $N_{N_G(Q)}(R)$-module.

Let g be an element of $N_G(QR)$ such that $N_{N_P(Q)}(R) \leq {}^gN_P(QR) \cap N_{N_G(Q)}(R)$. Then we have $Q^g \leq (QR)^g = QR \leq P$ and hence there is $h \in N_G(P)$ such that $gh^{-1} \in C_G(Q)$ since $\mathcal{F} = \mathcal{F}_P(N_G(P))$. We have that

$$\begin{aligned}
|N_{N_P(Q)}(R)| &\leq |{}^gN_P(QR) \cap N_{N_G(Q)}(R)| \\
&= |{}^gP \cap N_G(QR) \cap N_G(Q) \cap N_G(R)| \\
&= |{}^gP \cap N_G(Q) \cap N_G(R)| \\
&= |P \cap N_G(Q^g) \cap N_G(R^g)| \\
&= |N_{N_P(Q^g)}(R^g)| \\
&= |N_{N_P(Q^h)}(R^g)| \\
&= |N_{N_P(Q)^h}(R^g)| \\
&= |N_{N_P(Q)}(R^{gh^{-1}})^h| \\
&= |N_{N_P(Q)}(R^{gh^{-1}})|.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
R^{gh^{-1}} &\leq N_{N_P(Q)}(R)^{gh^{-1}} \\
&\leq ({}^gN_P(QR) \cap N_{N_G(Q)}(R))^{gh^{-1}} \\
&\leq ({}^gP \cap N_G(Q))^{gh^{-1}} \\
&= P^{h^{-1}} \cap N_G(Q^{gh^{-1}}) \\
&= P \cap N_G(Q) \\
&= N_P(Q)
\end{aligned}$$

and $gh^{-1} \in C_G(Q) \leq N_G(Q)$, the conjugation map $(\ )^{gh^{-1}} \colon R \to R^{gh^{-1}}$ is an isomorphism in $N_{\mathcal{F}}(Q)$. Since R is fully $N_{\mathcal{F}}(Q)$-normalized, we have that $|N_{N_P(Q)}(R^{gh^{-1}})| \leq |N_{N_P(Q)}(R)|$. Therefore, $N_{N_P(Q)}(R) = {}^gN_P(QR) \cap N_{N_G(Q)}(R)$, and $N_{N_P(Q)}(R)$ is maximal in $N_P(QR) \cap_{N_G(QR)} N_{N_G(Q)}(R)$, as desired. $\square$

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